Asymptotic approximation of the number of classic magic squares

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Number of Classic magic squares of order m: N(m)

Number of magic Series of classic magic squares of order m: S(m)

c-Factor of classic magic squares of order m: c(m)

Probability for a line (m disjoint numbers) to be magic: p(m)

$$N(m) = c(m) \cdot p(m)^{2m} \cdot \tfrac{1}{8} \cdot m^2!$$

$$p(m) = \frac{S(m)}{\binom{m^2}{m}}$$

Asymtotic approximations (very large m):

$$S(m) \simeq \tfrac{1}{\pi}\sqrt{\tfrac{3}{e}} \cdot \tfrac{(em)^m}{m^3} \cdot \left(1 + \tfrac{3}{5m}\right)$$

According to Henry Bottomley [1] and Michael Quist [2].

$$\binom{m^2}{m} \simeq \tfrac{(em)^m}{\sqrt{2\pi e m}} \cdot e^{\frac{1}{4m}}$$

See Proof 1.

$$p(m) \simeq \frac{S(m)}{\binom{m^2}{m}} \simeq \frac{\frac{1}{\pi}\sqrt{\frac{3}{e}} \cdot \frac{(em)^m}{m^3}}{\frac{(em)^m}{\sqrt{2\pi e m}}} \cdot \frac{1+\frac{3}{5m}}{e^{\frac{1}{4m}}} \Rightarrow \qquad \boxed{p(m) \simeq \sqrt{\frac{6}{\pi} \cdot m^{-5}} \cdot \frac{1+\frac{3}{5m}}{e^{\frac{1}{4m}}}}$$

$$N(m) \simeq c(m) \cdot \left(\sqrt{\frac{6}{\pi} \cdot m^{-5}} \cdot \frac{1+\frac{3}{5m}}{e^{\frac{1}{4m}}}\right)^{2m} \cdot \frac{1}{8} \cdot \sqrt{2\pi m^2}\left(e^{-1} \cdot m^2\right)^{m^2}$$

$$\boxed{N(m) \simeq \tfrac{1}{8} \cdot \sqrt{2\pi} \cdot e^{\frac{7}{10}} \cdot c(m) \cdot \left(\tfrac{6}{\pi}\right)^m \cdot m^{2m^2-5m+1} \cdot e^{-m^2}}$$

See Proof 2.

The small factor $c(m) \approx 0.185 \cdot (m-1)$ is still an experimental result.

$$\frac{c(m)}{m} \simeq 0.185 \Rightarrow \qquad \boxed{N(m) \simeq 0.185 \cdot \tfrac{1}{8} \cdot \sqrt{2\pi} \cdot e^{\frac{7}{10}} \cdot \left(\tfrac{6}{\pi}\right)^m \cdot m^{2m^2-5m+2} \cdot e^{-m^2}}$$

$$\ln\big(N(m)\big) \simeq \ln\left(0.185 \cdot \frac{\sqrt{2\pi}}{8}\right) + \frac{7}{10} + m \cdot \ln\left(\frac{6}{\pi}\right) + (2m^2 - 5m + 2) \cdot \ln(m) - m^2$$

Probability P(m) for a classic number square to be magic:

$$P(m) \simeq c(m) \cdot \left(\sqrt{\tfrac{6}{\pi} \cdot m^{-5}} \cdot \frac{1+\frac{3}{5m}}{e^{\frac{1}{4m}}}\right)^{2m} \simeq e^{\frac{7}{10}} \cdot 0.185m \cdot \left(\tfrac{6}{\pi}\right)^m \cdot m^{-5m}$$

Proof 1

Stirling formula: $n! = \sqrt{2\pi n} \cdot n^n \cdot e^{-n} \cdot e^{\frac{1}{12n}}$

$$\binom{m^2}{m} = \frac{m^2!}{m! \cdot (m^2 - m)!}$$

$$\binom{m^2}{m} \simeq \frac{\sqrt{2\pi m^2}}{\sqrt{2\pi m} \cdot \sqrt{2\pi(m^2 - m)}} \cdot \frac{(m^2)^{m^2}}{m^m \cdot (m^2 - m)^{m^2 - m}} \cdot \frac{e^{-m^2}}{e^{-m} \cdot e^{-m^2 + m}} \cdot \frac{e^{\frac{1}{12m^2}}}{e^{\frac{1}{12m}} \cdot e^{\frac{1}{12(m^2 - m)}}}$$

$$\binom{m^2}{m} \simeq \frac{1}{\sqrt{2\pi(m-1)}} \cdot \frac{(m^2 - m)^m}{m^m} \cdot \frac{(m^2)^{m^2}}{(m^2 - m)^{m^2}} \cdot e^{-\frac{1}{12m} - \frac{1}{12m^2(m-1)}}$$

$$\binom{m^2}{m} \simeq \frac{m^m}{\sqrt{2\pi(m-1)}} \cdot \left(\frac{m-1}{m}\right)^m \cdot \left(\left(\frac{m}{m-1}\right)^m\right)^m \cdot e^{-\frac{1}{12m}}$$

$$\binom{m^2}{m} \simeq \frac{m^m}{\sqrt{2\pi(m-1)}} \cdot e^{-1-\frac{1}{2m}} \cdot \left(e^{1+\frac{1}{2m}+\frac{1}{3m^2}}\right)^m \cdot e^{-\frac{1}{12m}}$$

$$\binom{m^2}{m} \simeq \frac{(em)^m}{\sqrt{2\pi e(m-1)}} \cdot e^{-\frac{1}{2m}} \cdot e^{+\frac{1}{3m}} \cdot e^{-\frac{1}{12m}}$$

$$\binom{m^2}{m} \simeq \frac{(em)^m}{\sqrt{2\pi em}} \cdot e^{-\frac{1}{4m}} \cdot \sqrt{\left(\frac{m}{m-1}\right)^{m \cdot \frac{1}{m}}}$$

$$\binom{m^2}{m} \simeq \frac{(em)^m}{\sqrt{2\pi em}} \cdot e^{-\frac{1}{4m}} \cdot e^{\frac{1}{2m}}$$

$$\binom{m^2}{m} \simeq \frac{(em)^m}{\sqrt{2\pi em}} \cdot e^{\frac{1}{4m}}$$

Proof 2

$$\left(\frac{1 + \frac{3}{5m}}{e^{\frac{1}{4m}}}\right)^{2m} = e^{-\frac{1}{2}} \cdot e^{2m \cdot \ln\left(1 + \frac{3}{5m}\right)} = e^{-\frac{1}{3}} \cdot e^{2m \cdot \left(\frac{3}{5m} - \frac{3}{10m^2} + \cdots\right)}$$

$$\left(\frac{1 + \frac{3}{5m}}{e^{\frac{1}{4m}}}\right)^{2m} \simeq e^{-\frac{1}{2}} \cdot e^{\frac{6}{5}} \simeq e^{\frac{7}{10}}$$